MATH 504 HOMEWORK 2

Due Friday, February 5.

Problem 1.

(1) *Suppose* $A \neq \emptyset$ *and there is a one-to-one function* $f : A \to B$*. Show that there is a surjective (i.e. onto) function* $g : B \to A$*.*

(2) *Suppose* B *can be well-ordered and there is a surjective function* $g : B \to A$*. Show that there is a one-to-one function* $f : A \to B$*.*

Problem 2. *In* ZF^- *prove the Schröder-Bernstein theorem i.e. that if* $A \preceq B$ *and* $B \preceq A$ *implies that* $A \approx B$*.*
Hint: Suppose $f : A \to B$ *and* $g : B \to A$ *are one-to-one.Set* $A_0 = A$, $B_0 = B$, $A_{n+1} = g\text{"}B_n$, $B_{n+1} = f\text{"}A_n$, $A_\infty = \bigcap_n A_n$, $B_\infty = \bigcap_n B_n$*. Let* $h(x)$ *be* $f(x)$ *if* $x \in A_\infty \cup \bigcup_n (A_{2n} \setminus A_{2n+1})$*. Otherwise let* $h(x)$ *be* $g^{-1}(x)$*. Show that* h *is well defined and* $h : A \to B$ *is one-to-one and onto.*

Problem 3. *Show that for infinite cardinals* $\kappa \geq \lambda$*,*

$$|\{X \subset \kappa : |X| = \lambda\}| = \kappa^\lambda.$$

Problem 4. *Let* λ *be an infinite cardinal and* κ *be any cardinal. Show that*

$$\kappa^{<\lambda} = \sup\{\kappa^\theta \mid \theta < \lambda, \theta \text{ is a cardinal}\}.$$

Problem 5. *Assume CH (but not GCH). Show that for every natural number* $n > 0$, $\omega_n^\omega = \omega_n$*.*